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Some rationalizability results for dynamic games

Ken-Ichi Akao,* Tapan Mitra† and Gerhard Sorger‡

We study the relation between dynamical systems describing the equilibrium behavior in dynamic games and those resulting from (single-player) dynamic optimization problems. More specifically, we derive conditions under which the dynamics generated by a model in one of these two classes can be rationalized by a model from the other class. We study this question under different assumptions about which fundamentals (e.g. technology, utility functions and time-preference) should be preserved by the rationalization. One interesting result is that rationalizing the equilibrium dynamics of a symmetric dynamic game by a dynamic optimization problem that preserves the technology and the utility function requires a higher degree of impatience compared to that of the players in the game.

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1 Introduction

Because rationality is one of the central hypotheses of modern economics, many economic models are formulated as optimization problems or as games. The key difference between these two frameworks is that the former describes the behavior of a single decision-maker whereas the latter takes into account the strategic behavior of several interacting agents. An interesting question then is whether these two classes of models generate results that are observationally distinguishable. We address this question in a dynamic context. More specifically, we define a rather general class of dynamic games that contains the class of dynamic optimization problems, and we study whether the dynamics generated by a dynamic game can also be generated by a (simpler) dynamic optimization problem. Therefore, the paper contributes to the economic literature on intertemporal inverse problems or, as they are also called, rationalizability problems. In that literature, one seeks to find conditions under which a given outcome (e.g. a dynamical system) can be represented as the solution

† Department of Economics, Cornell University, Ithaca NY, USA.

‡ Department of Economics, University of Vienna, Vienna, Austria.

* School of Social Sciences, Waseda University, Tokyo, Japan. Email: akao@waseda.jp

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of an economically meaningful model that satisfies certain assumptions. As for dynamic optimization models satisfying the standard assumptions of optimal growth theory, this literature is quite extensive. The pioneering works are Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986), and a survey is given in Sorger (2006).

One of the main motivations for the present study is that dynamic optimization models are far better understood than dynamic games. If it is possible to show that the equilibrium dynamics of dynamic games can be represented as optimal solutions of dynamic optimization problems, then one can apply the powerful apparatus of dynamic optimization theory to analyze the structure of equilibria of games. The rationalization of equilibrium dynamics by dynamic optimization problems can also provide interesting additional insights, if the fundamentals of the dynamic optimization problem are related in an unambiguous way to those of the underlying game. We will illustrate this by means of examples in which equilibrium dynamics generated by a game can only be rationalized by a dynamic optimization problem with the same specification of technology and utility function if the discount factor in the dynamic optimization problem is smaller than that in the game. In other words, the distortions caused by strategic interactions in the dynamic game are reflected by greater impatience in the rationalizing dynamic optimization problem.

The paper proceeds as follows. In Section 2, we introduce the notation and define the class of dynamic games. The games belonging to this class have a natural interpretation as common property resource models but, due to the generality of the specification, many other applications are also possible. The equilibrium concept we are using is strict Markov-perfect Nash equilibrium (MPNE), which is probably the most popular equilibrium concept for dynamic games of this form. Dynamic optimization problems are defined as special games with a single player. This class of dynamic optimization problems basically coincides with the class of optimal growth models that has been studied extensively in the rationalizability literature; see Stokey and Lucas (1989) for extensive coverage of these models or Sorger (2006) for their use in rationalizability questions.

In Section 3, we briefly discuss the problem of representing optimal solutions of dynamic optimization problems as equilibria of dynamic games. Although this problem seems trivial (because the class of dynamic games contains all dynamic optimization problems), we show that it is not if we impose certain properties on the structure of the game (e.g. by fixing the number of players), on the form of the equilibria (e.g. symmetry) or on the fundamentals that should be preserved in the transition from the dynamic optimization model to the dynamic game (e.g. technology and time-preference).[1]

In Section 4, we turn to the main topic of the paper; namely, to the problem of rationalizing the equilibrium dynamics of a dynamic game by a (single player) dynamic optimization problem. This problem is more challenging than that in Section 3, because the class of dynamic games is much larger than the class of dynamic optimization problems. The problem is also more relevant as it broadens the scope of the representative agent model. We first show that equilibria of dynamic games can exhibit phenomena that cannot occur in dynamic optimization problems satisfying standard assumptions. Despite this fact,

[1] Dana and Montrucchio (1986) address a similar problem and restrict the class of games to duopoly games in which the players make alternating moves.

we are able to prove a rationalizability result in a rather general framework. A noteworthy feature of this result is that both the technology and the time-preference are preserved by the rationalization. We use the famous "great fish war" game from Levhari and Mirman (1980) to illustrate how the requirement that certain fundamentals are preserved by a rationalization imposes unambiguous relations between other fundamentals. Indeed, any dynamic optimization problem that rationalizes the equilibrium dynamics of the "great fish war" using the same specification of the technology must necessarily also preserve the form of the utility function and it must necessarily have a smaller discount factor. We also discuss an example from growth theory in which the equilibrium dynamics of a game can only be rationalized by a corresponding representative agent model if the time-preference factor of the representative agent is smaller than the common time-preference factor of the players in the game. In the last part of Section 4, we show that this relation between the discount factors of dynamic games and the corresponding dynamic optimization problems that we observed in the examples must also hold in a somewhat broader context of games with isoelastic utility functions.

Section 5 contains concluding remarks. All proofs are presented in Section 6.

2 Definitions and preliminaries

In this section we formally define the class of dynamic games and dynamic optimization problems under consideration and we explain the concept of rationalizability. Time evolves in discrete periods; that is, the time variable t takes values in the set of positive integers $\mathbb{N}$. Furthermore, we denote by $x_{t-1} \in X$ the vector of state variables at the end of period $t-1$ (or, equivalently, at the beginning of period t). The state space X is assumed to be a non-empty subset of $\mathbb{R}^m_+$, the non-negative orthant of m-dimensional Euclidean space.

The technology of the economy is described by the transition possibility set $\Omega \subseteq X \times X$ and by the return function $R : \Omega \mapsto \mathbb{R}_+$. The transition possibility set Ω contains all pairs $(x, y) \in X \times X$ such that a transition from state x to state y is possible within one period. For each $x \in X$, we call $\Omega_x = \{y \in X \mid (x, y) \in \Omega\}$ the x-section of Ω, and we assume that Ω_x is non-empty for all $x \in X$. A state trajectory $(x_{t-1})_{t=1}^{\infty}$ is feasible if it satisfies $x_t \in \Omega_{x_{t-1}}$ for all $t \in \mathbb{N}$. The value of the return function, $R(x, y)$, denotes the amount of output that is available for consumption in a given period during which the state moves from x to y.

The economy is populated by n players, where $n \in \mathbb{N}$. All players are infinitely-lived and identical. We denote by c^i_t player i's consumption level in period t. Given a feasible state trajectory $(x_{t-1})_{t=1}^{\infty}$, the vector of individual consumption levels $(c^1_t, c^2_t, \ldots, c^n_t) \in \mathbb{R}^n_+$ is feasible in period t, if it satisfies $\sum_{i=1}^{n} c^i_t \leq R(x_{t-1}, x_t)$ for all $t \in \mathbb{N}$. Player $i \in \{1, 2, \ldots, n\}$ seeks to maximize the objective functional

$$\sum_{t=1}^{\infty} \rho^{t-1} u\left(c^i_t, x_{t-1}\right),$$

where $\rho \in (0, 1)$ is the discount factor and $u : \mathbb{R}_+ \times X \mapsto \mathbb{R}$ is the utility function, which may depend on the player's own consumption as well as on the state of the economy.

The game under consideration is described by the fundamentals $(n, X, \Omega, R, u, \rho)$. It is a simultaneous-move game with perfect information. Each player i adopts a stationary Markovian strategy $\sigma^i : X \mapsto \mathbb{R}_+$ that determines the individual consumption level in each period t as a function of the state of the system at the beginning of that period; that is, $c_t^i = \sigma^i(x_{t-1})$. A strategy profile $(\sigma^1, \sigma^2, \ldots, \sigma^n)$ is feasible if, for any initial state $x \in X$, there is a feasible path $(x_{t-1})_{t=1}^{\infty}$ that satisfies $x_0 = x$ and $\sum_{i=1}^{n} \sigma^i(x_{t-1}) \leq R(x_{t-1}, x_t)$ for all $t \in \mathbb{N}$.

A strategy profile $(\sigma^1, \sigma^2, \ldots, \sigma^n)$ is a *strict* MPNE if it is feasible and if, for each possible initial state $x \in X$ and for each player i, the dynamic optimization problem

$$\max_{\left(c_t^i, x_{t-1}^i\right)_{t=1}^{\infty}} \left\{ \liminf_{T\to\infty} \sum_{t=1}^{T} \rho^{t-1} u(c_t^i, x_{t-1}^i) \right\}$$
$$\text{subject to } \left(x_{t-1}^i, x_t^i\right) \in \Omega \quad \text{for all } t \in \mathbb{N},$$
$$R(x_{t-1}^i, x_t^i) - \sum_{j\neq i} \sigma^j(x_{t-1}^i) - c_t^i \geq 0 \quad \text{for all } t \in \mathbb{N}, x_0^i = x \tag{1}$$

has a unique solution $(c_t^{i*}, x_{t-1}^{i*})_{t=1}^{\infty}$ that satisfies the following two conditions:

(i) $c_t^{i*} = \sigma^i(x_{t-1}^{i*})$ for all $t \in \mathbb{N}$.

(ii) There exists a feasible path $(x_{t-1})_{t=1}^{\infty}$ with initial state $x_0 = x$ such that $x_t^{i*} = x_t$ holds for all $i \in \{1, 2, \ldots, n\}$ and all $t \in \mathbb{N}$.

The strict MPNE $(\sigma^1, \sigma^2, \ldots, \sigma^n)$ is symmetric if there exists a function $\sigma : X \to \mathbb{R}_+$ such that $\sigma^i = \sigma$ holds for all $i \in \{1, 2, \ldots, n\}$. We say that a function F: $X \to X$ is *rationalized* by a dynamic game $(n, X, \Omega, R, u, \rho)$ if, for all $x_0 \in X$, it holds that the game $(n, X, \Omega, R, u, \rho)$ with initial state x_0 has a strict MPNE generating a state trajectory $(x_{t-1})_{t=1}^{\infty}$ that satisfies the difference equation $x_t = F(x_{t-1})$.

A dynamic optimization problem is a special case of a dynamic game in which the number of players, n, is equal to 1. The goal of the present paper is to determine under which assumptions the equilibrium dynamics that are generated by a dynamic game can be rationalized by a dynamic optimization problem, and vice versa.

3 Rationalization by a dynamic game

Because a dynamic optimization problem is a special case of a dynamic game with a single player, it is obvious that the dynamics generated by a dynamic optimization problem can always be rationalized by a dynamic game. This finding, however, becomes non-trivial if we impose additional restrictions on the game (e.g. the number of players) or on the form of the equilibrium (e.g. symmetry). Such a result is the content of the present section and is included here for completeness and to set the stage for the material presented later.

To get started, let us point out that the dynamics generated by the dynamic optimization problem (i.e. a single-player dynamic game) $(1, X, \Omega, R, u, \rho)$ in which the utility function u

is strictly increasing in consumption coincides with the dynamics generated by the problem $(1, X, \Omega, \tilde{R}, \mathrm{p}_c, \rho)$, where $\mathrm{p}_c(c, x) = c$ and $\tilde{R}(x, y) = u(R(x, y), x)$. In words, in the case of dynamic optimization problems, the different roles played by the return function R and the utility function u are not really identified. In this section, we shall start from a model of the form $(1, X, \Omega, R, \mathrm{p}_c, \rho)$ and we will simplify the presentation by dropping $n = 1$ and $u = \mathrm{p}_c$ from the notation. Hence, a dynamic optimization problem will be denoted by (X, Ω, R, ρ) and it takes the form

$$\begin{aligned} &\max_{(x_t)_{t=1}^{\infty}} \left\{ \liminf_{T\to\infty} \sum_{t=1}^{\infty} \rho^{t-1} R(x_{t-1}, x_t) \right\} \\ &\text{subject to } (x_{t-1}, x_t) \in \Omega \quad \text{for all } t \in \mathbb{N}. \end{aligned} \tag{2}$$

We denote by $V : X \mapsto \mathbb{R}_+ \cup \{\infty\}$ the optimal value function of (X, Ω, R, ρ). It is known that this function satisfies the Bellman equation

$$V(x) = \max_y \{ R(x, y) + \rho V(y) \mid (x, y) \in \Omega \}. \tag{3}$$

If it is true for all $x \in X$ that the problem on the right-hand side of (3) has a unique solution, say, $y = h(x)$, then we call the function h: $X \to X$ the optimal policy function of (X, Ω, R, ρ). Note that h is the optimal policy function of (X, Ω, R, ρ) if and only if h is rationalized by (X, Ω, R, ρ). We are now ready to state the main result of the present section.[2]

Theorem 1 *Let (X, Ω, R, ρ) be a dynamic optimization problem with optimal value function V and optimal policy function h, where h: $X \mapsto X$ is a given function and $V(x) < \infty$ for all $x \in X$. For every $n \geq 2$there exists a utility function $u : \mathbb{R}_+ \times X \mapsto \mathbb{R}$such that the dynamic game $(n, X, \Omega, R, u, \rho)$ rationalizes h through a symmetric strict MPNE.*

The strength of Theorem 1 derives from two facts. First, symmetry of the equilibrium has been imposed and, second, we have allowed ourselves only very little freedom for the construction of the dynamic game. This is the case because we have fixed the number of the players n, and because the rationalization preserves the transition possibility set Ω, the return function R, and the discount factor ρ.[3] This means that the only degree of freedom in the specification of the dynamic game $(n, X, \Omega, R, u, \rho)$ concerns the utility function u.

Theorem 1 can be used to obtain explicit solutions for dynamic games, based on explicit solutions to representative agent problems. While explicit solutions to the latter class of problems are not abundant, explicit solutions to the former class of problems are rare in

[2] All proofs can be found in Section 6.

[3] If we allow the technology R to change, the rationalization in both directions is rather trivial: (i) The optimal policy function h of a dynamic optimization problem $(1, X, \Omega, R, u, \rho)$ with the optimal consumption policy σ is rationalized by the dynamic game $(n, X, \Omega, \tilde{R}, u, \rho)$ with $\tilde{R}(x, y) = R(x, y) + (n-1)\sigma(x)$ for any $n \geq 2$. (ii) The equilibrium policy function h of the symmetric dynamic game $(n, X, \Omega, R, u, \rho)$ with a symmetric strategy σ is rationalized by the optimal dynamic optimization problem $(1, X, \Omega, \tilde{R}, u, \rho)$ with $\tilde{R}(x, y) = R(x, y) - (n-1)\sigma(x)$. These observations were made by a referee.

comparison. We conclude this section by illustrating this remark with Weitzman's example of a representative agent problem, as reported in Samuelson (1973).

Let the state space be $X = [0, 1]$ and let the transition possibility set Ω be defined as $\Omega = X \times X$. Next, define $R : \Omega \mapsto \mathbb{R}_+$ by

$$R(x, z) = x^{1/2}(1 - z)^{1/2} \quad \text{for all } (x, z) \in \Omega.$$

Finally, choose $\rho \in (0, 1)$. Then, we have the dynamic optimization problem (X, Ω, R, ρ).[4] It is known that the function h defined by

$$h(x) = \frac{\rho^2(1 - x)}{x + \rho^2(1 - x)} \quad \text{for all } x \in X \tag{4}$$

is the optimal policy function of this dynamic optimization problem Thus, the socially optimal solution generates the dynamics

$$x_{t+1} = h(x_t) \quad \text{for } t \geq 0 \tag{5}$$

given any $x_0 \in X$.

Now suppose we consider the same environment with $n = 2$ identical players. The proof of Theorem 1 tells us that if we define the utility function of each player as:

$$u(c, x) = (c/2) + (1/4)R(x, h(x)), \tag{6}$$

where h is given by (4), then the two-person dynamic non-cooperative game $(2, X, \Omega, R, u, \rho)$ also generates the dynamics given by (5).

Since R is decreasing in z, and $h(x)$ is decreasing in x, $R(x, h(x))$ is increasing in x both because of the first argument of R and because of the second argument of R. One thus has a *positive* stock effect in the utility function equation (6). This can be interpreted in the following way. The non-cooperative dynamic game can produce the same dynamics as the representative agent's (or social planner's) problem if each player has an appropriate *conservationist motive* (reflected in a positive stock effect). This positive stock effect compensates for the typical tendency to consume more of the resource in a dynamic game than in the social planner's problem, when the period utility function (as well as the discount factor and technology) is the same in both cases.

4 Rationalization by a dynamic optimization problem

We now turn to the problem of rationalizing the equilibrium dynamics of a dynamic game by a dynamic optimization problem. This problem is more complicated because the class

[4] We are presenting the model in a somewhat different way than Samuelson (1973). Thus, in our formulation, $R(x, z)$ is the amount of output available for consumption when the state moves from x to z, and the utility from consumption is measured by the consumption itself. In Samuelson's formulation, $R(x, z)$ is the reduced-form utility function, obtained from a primitive utility function based on the consumption of two goods, bread and wine, and x (respectively, z) is the amount of labor allocated today (respectively, tomorrow) to the production of a third good, grape juice, which through a process of aging becomes wine in one period.

of dynamic games is more general than the class of dynamic optimization problems. To appreciate this point consider the class of dynamic optimization problems (X, Ω, R, ρ) for which X and Ω are convex sets and R is a strictly concave function. It is well known that the optimal policy function of any such problem is continuous on X. However, there exist dynamic games $(n, X, \Omega, R, u, \rho)$ for which X, Ω and R satisfy the abovementioned convexity assumptions, for which u is a strictly increasing and strictly concave function of consumption, and for which the equilibrium dynamics are given by $x_{t+1} = F(x_t)$, where F is a discontinuous function; see Dutta and Sundaram (1993) for an example. This clearly demonstrates that equilibria of dynamic games can exhibit phenomena that are ruled out in dynamic optimization problems. Nevertheless, it is possible to prove a partial converse of Theorem 1, and we shall do so in the first part of this section. Thereafter, we shall show in the context of two examples that rationalizing the equilibrium dynamics by a dynamic optimization problem featuring the same technology and utility function as the original game implies that the discount factor of the optimization problem is smaller than that of the game.[5] We will conclude the section by presenting a result that generalizes the latter finding to a wider class of games.

4.1 A partial converse of Theorem 1

Let $(n, X, \Omega, R, u, \rho)$ be a dynamic game and let $(\sigma, \sigma, \ldots, \sigma)$ with $\sigma : X \mapsto \mathbb{R}_+$ be a symmetric strict MPNE of that game. We impose the following assumptions.

Assumption 1 (a) *The instantaneous utility function* $u : \mathbb{R}_+ \times X \mapsto \mathbb{R} \cup \{-\infty\}$ *is strictly increasing and concave with respect to its first argument (i.e. with respect to consumption). If* $c > 0$*, it holds that* $u(c, x) > -\infty$ *for all* $x \in X$*.*
(b) *The equilibrium value function* $v : X \mapsto \mathbb{R} \cup \{-\infty\}$ *defined by*

$$v(x) = \max_{(c_t, x_{t-1})_{t=1}^{\infty}} \sum_{t=1}^{\infty} \rho^{t-1} u(c_t, x_{t-1})$$

$$\textit{subject to} \quad (x_{t-1}, x_t) \in \Omega \quad \textit{for all } t \in \mathbb{N},$$

$$R(x_{t-1}, x_t) - (n-1)\sigma(x_{t-1}) - c_t \geq 0 \quad \textit{for all } t \in \mathbb{N},$$

$$x_0 = x$$

satisfies

$$-\infty < v(x) < \infty \quad \textit{for all } x \in X \setminus \{0\}. \tag{7}$$

(c) *For each* $x \in X$ *there is a unique value* $h(x) \in \Omega_x$ *such that*

$$\begin{aligned} v(x) &= u(\sigma(x), x) + \rho v(h(x)) \\ &= \max_{c \in \mathbb{R}_+, y \in \Omega_x} \{u(c, x) + \rho v(y) \mid R(x, y) - (n-1)\sigma(x) - c \geq 0\}. \end{aligned} \tag{8}$$

[5] One of these examples is the "great fish war" from Levhari and Mirman (1980). This is the same game that underlies the example of discontinuous equilibrium dynamics presented in Dutta and Sundaram (1993).

Assumption 2 *Ω is a non-empty and convex set and $R : \Omega \mapsto \mathbb{R}_+$ is a concave function.*

Assumption 3 *It holds that $\Omega_0 = \{0\}$, $\Omega_x \subseteq \Omega_y$ for all $x, y \in X$ such that $x \leq y$, and* $R(0, 0) = 0$. *Moreover, $R(x, y)$ is increasing in x and decreasing in y. If $x \neq 0$, then there exists $y \in \Omega_x$ such that* $R(x, y) > 0$.

Assumption 4 *$v : X \mapsto \mathbb{R} \cup \{-\infty\}$ is a concave function.*

Assumption 5 *For each $x \in X \backslash \{0\}$, the transversality condition*

$$\lim_{t\to\infty} \rho^t p_t h^t(x) = 0$$

holds, where $p_t \in \partial v(h^t(x))$ and $\partial v(x)$ is the subdifferential of v at $x \in X$.[6]

Note that the equilibrium dynamics generated by a symmetric strict MPNE of the game $(n, X, \Omega, R, u, \rho)$ are described by the dynamical system $x_t = h(x_{t-1})$, where h: $X \mapsto X$ is implicitly defined in Assumption 1(c). The following theorem shows that these dynamics can also be rationalized by a dynamic optimization problem.

Theorem 2 *Let $(n, X, \Omega, R, u, \rho)$ be a dynamic game satisfying Assumptions 1–5. There exists a function $U : \mathbb{R}_+ \times X \mapsto \mathbb{R} \cup \{-\infty\}$ such that h is the optimal policy function of the dynamic optimization problem* $(1, X, \Omega, R, U, \rho)$.

To appreciate the relevance of Theorem 2 it is necessary to understand the underlying assumptions. Assumption 1 imposes restrictions on the dynamic game $(n, X, \Omega, R, u, \rho)$ that correspond one-to-one to those that were imposed on the dynamic optimization problem (X, Ω, R, ρ) in Theorem 1. Assumptions 2 and 3 are quite innocuous standard assumptions that hold in many economic applications.[7] The only critical assumption of Theorem 2 is the concavity of v stated in Assumption 4. Indeed, since the equilibrium strategies in a MPNE are typically nonlinear functions of the state variables, the concavity of the equilibrium value functions does not follow from the convexity of the feasible sets and the concavity of the utility functions.[8] In contrast, for a dynamic optimization problem, these convexity properties imply the concavity of the optimal value function. Because the proof of Theorem 2 is based on the construction of a dynamic optimization problem with the same value function as the given dynamic game, we must assume that this function is concave. It is also worth mentioning that the example from Dutta and Sundaram (1993)

[6] A sufficient condition for the necessity of the transversality condition is given by Kamihigashi (2003, theorem 2.2).

[7] As can be seen from the proof of Theorem 2, these assumptions may also be replaced by others as far as they ensure that the optimization problem in equation (8) is a convex one and that Slater's constraint qualification is satisfied.

[8] Nevertheless, there exist symmetric MPNE of dynamic game models that satisfy Assumptions 1–5. Examples are the equilibrium of the great fish war model discussed by Levhari and Mirman (1980), which is studied in the next subsection, and the sustained Nash equilibrium growth model, which is studied in Subsection4.3. It must also be emphasized that Assumptions 1–5 are imposed on a given strict MPNE. It can be the case that a game has an MPNE for which these assumptions hold and another MPNE for which they are violated.

that we referred to at the outset of this section fails to satisfy Assumption 4, which gives rise to the discontinuity of the equilibrium dynamics.

Next let us make a few comments on the theorem itself. An important point here is that the rationalization by a dynamic optimization problem preserves the technology (as described by the transition possibility set Ω and the return function R) as well as the discount factor ρ. In other words, we can interpret U as the utility function of a hypothetical representative agent who has the same time-preference as the players of the game and who generates the equilibrium dynamics of the MPNE. Rationalizing the equilibrium dynamics of a dynamic game by a dynamic optimization problem broadens the scope of the representative agent model, because one can keep track of the dynamic game solution by studying the more familiar representative agent model.[9]

As shown in the Appendix, our proof of the existence theorem proceeds by actually defining a particular utility function of the representative agent and checking that it rationalizes the path generated by a symmetric MPNE of the dynamic game. The utility function that we use can be given the following interpretation. A common social welfare function W used to measure the well-being of society (consisting of n individuals with identical utility function u) in a period is to evaluate the utility of the typical individual if the aggregate consumption of society is distributed equally among the n individuals. Thus, $W(c, x) = u(c/n, x)$; this is the average utilitarian social welfare function. Our utility function of the representative agent $U(c, x)$ differs from this benchmark $W(c, x)$ in that it puts more weight on average consumption (c/n). That is, given the stock x, the marginal utility from consumption is larger for our utility function U than for the welfare function W.[10]

It is important to note that the utility function, U, that we use is not proved to be the unique utility function that rationalizes the path generated by a symmetric MPNE of the dynamic game. Thus, it is not clear whether the above interpretation can always be given to a utility function that rationalizes the path generated by a symmetric MPNE of the dynamic game. We leave this as an interesting open question.

4.2 The Levhari–Mirman example

Theorem 2 shows that, under certain assumptions, it is possible to rationalize the equilibrium dynamics generated by a game through a dynamic optimization problem. Such a rationalization, however, is typically not unique. Therefore, in the present subsection, we want to reconsider the same question from a slightly different angle. In contrast to the previous subsection, we now consider a very specific game $(n, X, \Omega, R, u, \rho)$ in which the utility function u does not depend on the state variable, and we require that the equilibrium dynamics generated by that game are rationalized by a dynamic optimization problem that preserves the technology (Ω and R) but not necessarily the discount factor ρ. Moreover, we

[9] In the case where the policy function h is Lipschitz continuous, another rationalizability result follows immediately from theorem 3 of Mitra and Sorger (1999). However, the optimization problem that is used to prove the Mitra–Sorger theorem typically neither shares the technology nor the time-preference with the game, unlike the case in Theorem 2.

[10] The stock effect is not unambiguous because the equilibrium strategy and a Lagrange multiplier, both of which depend on the stock x, are used in our definition of U.

require that the utility function of the representative agent, U, also depends on consumption only. We will show that this has interesting implications for the utility function U and the discount factor δ of the corresponding dynamic optimization problem.

The example we want to study is the famous "great fish war" from Levhari and Mirman (1980). This game $(n, X, \Omega, R, u, \rho)$ is specified by $n = 2$, $X = [0, 1]$, $\Omega = \{(x, y) \mid 0 \leq y \leq x^{\alpha}\}$, $R(x, y) = x - y^{1/\alpha}$, $u(c, x) = \ln(c)$ and $\rho \in (0, 1)$. It is well known that a symmetric strict MPNE of this game is given by (σ, σ), where

$$\sigma(x) = \frac{(1 - \alpha\rho)x}{2 - \alpha\rho},$$

and that the state dynamics generated by this MPNE are given by $x_t = h(x_{t-1})$, where

$$h(x) = \left(\frac{\alpha\rho x}{2 - \alpha\rho}\right)^{\alpha}. \tag{9}$$

It is straightforward to verify that h is the optimal policy function of the dynamic optimization problem $(1, X, \Omega, R, u, \delta)$ with $\delta = \rho/(2 - \alpha\rho)$. Note that both the return function and the utility function of this dynamic optimization problem are the same as in the underlying dynamic game. Note, furthermore, that the discount factors of the two models, ρ and δ, respectively, do not coincide but that δ is smaller than ρ.

In the following theorem we show that the preferences (instantaneous utility function and discount factor) of any dynamic optimization problem that allows the rationalization of the game dynamics equation (9) are uniquely determined up to an increasing affine transformation of the instantaneous utility function as long as we impose that the technology is described by Ω and R. This uniqueness result provides us with useful information about the relationship between the parameters of the original dynamic game and the parameters of the surrogate representative agent model.

Theorem 3 *Suppose that the function h defined by* (9) *is rationalized by a dynamic optimization problem of the form* $(1, X, \Omega, R, U, \delta)$, *where* $U : [0, 1] \mapsto \mathbb{R} \cup \{-\infty\}$ *is increasing and strictly concave on* $[0, 1]$ *and twice continuously differentiable on the interior of its domain, and where* $\delta \in (0, 1)$. *Then it follows that* $\delta = \rho/(2 - \alpha\rho)$ *and that* U *is an increasing affine transformation of* u.

Theorem 3 shows that, to generate the equilibrium dynamics of the "great fish war" in a dynamic optimization problem, the decision-maker in the latter model must use (essentially) the same utility function as the agents in the dynamic game and he or she must have a higher degree of impatience compared to the players of the dynamic game.

Levhari and Mirman (1980) show that the cooperative solution of their model $(2, X, \Omega, R, u, \rho)$ generates the dynamics $x_t = \bar{h}(x_{t-1})$ with

$$\bar{h}(x) = (\alpha\rho x)^{\alpha}.$$

The resource stock is lower at each date t in the symmetric MPNE compared to the cooperative solution. Furthermore, the steady state in the former is lower than in the latter. The cooperative solution should not be confused with the dynamic optimization problem $(1, X, \Omega, R, U, \delta)$ from Theorem 3. The dynamics of the former differ from the equilibrium dynamics of the game, whereas the dynamic optimization problem from Theorem 3 produces by construction the same dynamics as the original game. However, the phenomenon of overexploitation in the symmetric MPNE (compared to the cooperative solution) is reflected precisely in the higher degree of impatience of the representative agent compared to that of the two players in the game.

The relation between the discount factor of the dynamic optimization problem, δ, and that of the dynamic game, ρ, implies a discount factor restriction for the dynamic optimization problem. Since $\delta = \rho/(2 - \alpha\rho)$ and $\rho \in (0, 1)$, it follows that $\delta < 1/(2 - \alpha)$ independently of the actual value of the common discount factor of the two players. In particular, even if the players are arbitrarily patient, that is, as ρ approaches 1, the representative agent's discount factor is uniformly bounded below 1.

4.3 Example of sustained Nash equilibrium growth

As a second example of rationalizability of a dynamic game solution by the solution of a representative agent model, we consider a framework in which it is possible for the economy to produce sustained growth of consumption by accumulating capital, and the typical agent's utility function depends both on the agent's own consumption and the aggregate capital stock of the economy (a wealth effect). We are interested in characterizing the path of *equilibrium growth*, where the notion of equilibrium is the typical one used in dynamic games; namely, a strict MPNE.

The *optimal growth* problem, in which utility is derived from consumption as well as the capital stock, was first studied (in the standard neoclassical model in which per-capita consumption remains bounded on all paths) in Kurz (1968). For a model in which unbounded growth of per-capita consumption is possible, Roy (2010) provides a comprehensive study of the optimal growth problem.

In the example that we study, the game $(n, X, \Omega, R, u, \rho)$ is specified by $n = 2$, $X = \mathbb{R}_+$, $\Omega = \{(x, y) \mid 0 \le y \le Ax\}$, $R(x, y) = Ax - y$, $u(c, x) = c^\alpha x^\beta$ and $\rho \in (0, 1)$. The parameters of the model are assumed to satisfy the following restrictions:

$$A > 1;\ \alpha + \beta < 1;\ \rho A^{\alpha+\beta} < 1.$$

It can be shown that a symmetric strict MPNE of this game is given by (σ, σ), where $\sigma(x) = \gamma Ax$ and where $\gamma \in (0, 1/2)$ is the unique solution to the following equation:

$$\frac{\alpha(1 - 2\gamma)^{1-\alpha-\beta}}{\rho A^{\alpha+\beta}\{(1 - \gamma)\alpha + \gamma\beta\}} = 1. \tag{10}$$

The dynamics of the capital stock generated by this MPNE are given by $x_t = h(x_{t-1})$, where

$$h(x) = (1 - 2\gamma)Ax \quad \text{for all } x \ge 0.$$

If one considers the representative agent's dynamic optimization problem $(1, X, \Omega, R, u, r)$ with $r \in (0, 1)$ satisfying $rA^{\alpha+\beta} < 1$, then it can be shown that $s(x) = \eta Ax$ is the optimal consumption policy function, where $\eta \in (0, 1)$ is the unique solution to the following equation

$$\frac{\alpha(1-\eta)^{1-\alpha-\beta}}{r A^{\alpha+\beta}\{\alpha+\eta\beta\}} = 1. \tag{11}$$

To rationalize the strict MPNE (σ, σ), one has to generate the path of capital stocks,

$$x_{t+1} = (1-2\gamma)Ax_t,$$

from every initial capital stock, $x_0 = x > 0$, in the representative agent model. This means that the optimal consumption function, $s(x)$, in the representative agent model must be such that $\eta = 2\gamma$. Since γ is given by (10) and η is given by (11), we must have

$$r = \rho\frac{\{\alpha+\gamma(\beta-\alpha)\}}{\{\alpha+2\gamma\beta\}} < \rho.$$

That is, to rationalize the strict MPNE (σ, σ), in the representative agent model, the representative agent must use a smaller discount factor than the players in the dynamic game.

4.4 A more general framework

In the previous two subsections we provide examples in which the equilibrium dynamics of a game can be rationalized by a dynamic optimization problem in such a way that the technology is preserved.[11] Both examples show that preserving the technology and the instantaneous utility function implies that the discount factor of the optimization problem is smaller than that of the game. In the Levhari and Mirman example we show that preserving the technology implies that the instantaneous utility function is also preserved. In the present subsection, we return to a more general framework with isoelastic instantaneous utility function and we assume that both the technology and the instantaneous utility function are preserved. Under these assumptions, we generalize the findings from the two examples by proving that the discount factor of the dynamic optimization problem must be smaller than that of the game.

Let $(n, X, \Omega, R, u, \rho)$ be a dynamic game with a continuously differentiable return function R. We denote the partial derivatives of R with respect to its two arguments by $R_1(x, y) = \partial R(x, y)/\partial x$ and $R_2(x, y) = \partial R(x, y)/\partial y$. Assume that $R_1(x, y) > 0$ for all $(x, y) \in \Omega$ and that the instantaneous utility function u is a concave and isoelastic function of consumption; that is, there exists $\eta > 0$ such that $-cu''(c)/u'(c) = \eta$ for all $c > 0$. Let $\sigma : X \mapsto \mathbb{R}$ be a symmetric MPNE strategy of the game and let h: $X \mapsto X$ be the function

[11] The example of sustained Nash equilibrium growth can be extended with instantaneous utility function $u(c, x) = [(x\phi(c/x))^{1-\eta} - 1]/(1-\eta)$, where ϕ is an increasing and concave function with $\phi(0) = 0$ and where $\eta > 0$. The instantaneous utility function includes $u(c, x) = c^\alpha x^\beta$ as a special case ($\phi(z) = z^{\alpha/(\alpha+\beta)}$ and $\eta = 1 - (\alpha+\beta)$). In the case where ϕ is the identity map, we have a standard model of economic growth with a linear technology, whereas the case where ϕ is strictly concave allows one to capture wealth effects as well.

that describes the associated equilibrium dynamics. By definition, these functions satisfy $\sigma(x) = n^{-1}R(x, h(x))$ for all $x \in X$.

Theorem 4 *Suppose that the function h is rationalized by the dynamic optimization problem* $(1, X, \Omega, R, u, \delta)$. *Let* $(x_t)_{t=0}^{\infty}$ *be a path generated by h such that* $x_t \in \operatorname{int}\Omega_{x_{t-1}}$ *for all* $t \in \mathbb{N}$. *If there exists a period* $t \geq 1$ *such that the equilibrium strategy* σ *is differentiable at* x_t *and* $\sigma'(x_t) > 0$, *then* $\rho > \delta$.

The above theorem indicates that if an outcome of a dynamic game is rationalized by a dynamic optimization problem involving a representative agent with the same instantaneous utility function as the players in the game, then the representative agent must be more impatient than the players of the original game. In other words, the inefficiency caused by the strategic interaction among players is translated into heavier discounting by the representative agent.

The following corollary considers the case where $\sigma'(\bar{x}) > 0$ holds at an optimal steady state $\bar{x}$.

Corollary 1 *Suppose that the function h is rationalized by the dynamic optimization problem* $(1, X, \Omega, R, u, \delta)$. *If there is an interior optimal steady state* $\bar{x} = h(\bar{x}) > 0$ *that is stable* $(|h'(\bar{x})| < 1)$, *then it follows that* $\rho > \delta$.

5 Concluding remarks

In the present paper we have investigated how the dynamics generated by dynamic optimization problems are related to those generated by dynamic games. More specifically, we have explored whether dynamics generated by models from one of these two classes can also be rationalized by models from the other class under the additional restriction that some of the fundamentals are preserved by the rationalization. Our results have very interesting implications about those fundamentals that are not preserved by the rationalization. For example, we have shown that a non-cooperative dynamic game can generate the same dynamics as a representative agent problem with the same technology and time-preference provided that the utility function of each player displays a positive stock effect. We have also shown that the equilibrium dynamics of a game can be rationalized by a representative agent problem with the same technology and utility function only if the representative agent discounts the future more heavily. These findings provide new interpretations of the effects of strategic interaction in dynamic settings.

6 Proofs

Proof of Theorem 1 From the assumptions stated in the theorem, for all $(x, y) \in \Omega$ with $y \neq h(x)$, the following holds:

$$\infty > V(x) = R(x, h(x)) + \rho V(h(x)) > R(x, y) + \rho V(y). \tag{12}$$

Moreover, since $R(x, y) \geq 0$ for all $(x, y) \in \Omega$, it follows that $V(x)$ is non-negative for all $x \in X$. Finally, the assumption $V(x) < \infty$ implies that

$$\lim_{t\to\infty} \rho^t V(h^t(x)) = 0 \quad \text{for all } x \in X. \tag{13}$$

This is the case because

$$\begin{aligned}\infty > V(x) &= \sum_{t=1}^{\infty} \rho^{t-1} R(h^{t-1}(x), h^t(x)) \\ &= \lim_{T\to\infty} \left(\sum_{t=1}^{T} \rho^{t-1} R(h^{t-1}(x), h^t(x)) + \rho^T V(h^T(x)) \right).\end{aligned}$$

We now proceed in two steps. The first step consists of the following lemma, in which u_1 denotes the partial derivative of u with respect to its first argument.

Lemma 1 *Let the assumptions of Theorem 1 be satisfied. Furthermore, let $n \geq 2$ be an arbitrary integer and assume that there exists a function u satisfying the following assumptions P_u1–P_u3:*

P_u1: For all $x \in X$, it holds that $u(c, x)$ is a strictly increasing and concave function of c.

P_u2: For all $x, \in X$, $u_1(n^{-1}R(x, h(x)), x)$ exists and satisfies $u_1(n^{-1}R(x, h(x)), x) = n^{-1}$.

P_u3: For all $x \in X$, it holds that $u(n^{-1}R(x, h(x)), x) = n^{-1}R(x, h(x))$.

Then, it follows that h is rationalized by the dynamic game$(n, X, \Omega, R, u, \rho)$ through a symmetric strict MPNE .

Proof 1: Suppose that the function u satisfies assumptions P_u1–P_u3 and that all players other than player i use the Markovian strategy $\sigma(x) = n^{-1}R(x, h(x))$. Then, player i's problem is written as

$$\begin{aligned}&\max_{(x_{t-1})_{t=1}^{\infty}} \left\{ \liminf_{T\to\infty} \sum_{t=1}^{T} \rho^{t-1} u\left(R(x_{t-1}, x_t) - (n-1)\,\sigma(x_{t-1}), x_{t-1}\right) \right\} \\ &\text{subject to } (x_{t-1}, x_t) \in \Omega \quad \text{for all } t \in \mathbb{N}, \\ &\qquad R(x_{t-1}, x_t) - (n-1)\,\sigma(x_{t-1}) \geq 0 \quad \text{for all } t \in \mathbb{N}.\end{aligned} \tag{14}$$

If h is the optimal policy function for this problem, then it follows that σ is the unique best response of player i, because $R(x_{t-1}, h(x_{t-1})) - (n-1)\sigma(x_{t-1}) = n^{-1}R(x_{t-1}, h(x_{t-1})) = \sigma(x_{t-1})$. This, in turn, implies that $(\sigma, \sigma, \ldots, \sigma)$ is a MPNE and that h is rationalized by the dynamic game $(n, X, \Omega, R, u, \rho)$. Therefore, all we need to show is that h is the optimal policy function of problem (14).

To this end, note that P_u 1, P_u2 and (12) imply for all $(x, y) \in \Omega$ that

$$\begin{aligned} & u\left(R(x, h(x)) - (n-1)\sigma(x), x\right) - u\left(R(x, y) - (n-1)\sigma(x), x\right) \\ & \quad \geq \{u_1\left(R(x, h(x)) - (n-1)\sigma(x), x\right)\}\left[R(x, h(x)) - R(x, y)\right] \\ & \quad = n^{-1}\left[R(x, h(x)) - R(x, y)\right] \\ & \quad \geq n^{-1}[\rho V(y) - \rho V(h(x))]. \end{aligned}$$

Therefore, for all $(x, y) \in \Omega$ it holds that

$$\begin{aligned} & u\left(R(x, h(x)) - (n-1)\sigma(x), x\right) + n^{-1}\rho V(h(x)) \\ & \quad \geq u\left(R(x, y) - (n-1)\sigma(x), x\right) + n^{-1}\rho V(y). \end{aligned} \tag{15}$$

Next, we note that assumption P_u3 and the definition of σ imply that

$$u\left(R(x, h(x)) - (n-1)\sigma(x), x\right) = n^{-1}R(x, h(x)). \tag{16}$$

It follows from this equality together with (12) and (15) that

$$n^{-1}V(x) = \max_y\{u\left(R(x, y) - (n-1)\sigma(x), x\right) + \rho n^{-1}V(y) \mid (x, y) \in \Omega\} \tag{17}$$

holds for all $x \in X$. Since $V(x)$ is bounded from below, the path $(h^t(x))_{t=0}^{\infty}$ is an optimal path for problem (14) starting from the initial state x, which is proved by applying the proof of lemma 3.1 in Dockner, Jørgensen, Long, and Sorger (2000). We have also shown that the optimal value function of that problem is $n^{-1}V(x)$ and that the corresponding Bellman equation is (17). It remains to be shown that the right-hand side of (17) has a unique maximizer. To this end note that (12) implies that (15) holds with strict inequality if $y \neq h(x)$. However, this obviously implies that the unique maximizer on the right-hand side of (17) is given by $h(x)$, which completes the proof of the lemma. □

The second step of the proof of Theorem 1 is to show that there exists a utility function u satisfying P_u1–P_u3. However, this is obvious as can be seen from the following examples:

$$u(c, x) = c^{1/n}[n^{-1}R(x, h(x))]^{(n-1)/n}$$

and

$$u(c, x) = \frac{1}{n}c + \frac{n-1}{n}[n^{-1}R(x, h(x))].$$

Proof of Theorem 2 The fact that u is strictly increasing in c implies that the inequality in (8) must hold as an equality if the consumption rate c equals its equilibrium value $\sigma(x)$. This implies that

$$\sigma(x) = n^{-1}R(x, h(x)). \tag{18}$$

Also note that

$$\lim_{t\to\infty} \rho^t v(h^t(x)) = 0 \quad \text{for all } x \in X \setminus \{0\} \tag{19}$$

must hold for all $x \in X$ because of the results in (7) and (8). When $x = 0$, Assumption 3 implies that the feasible consumption and the capital stock at the end of the period are uniquely given by $(c, y) = (0, 0)$. Therefore, the equilibrium strategy for this case must be $\sigma(0) = 0$ and $h(0) = 0$. However, if $x \neq 0$, then the maximization problem in (8) satisfies Slater's constraint qualification: $R(x, y) - (n-1)\sigma(x) - c$ is concave in (c, y) and its value is strictly positive at some (c, y) such that $0 \leq y \leq h(x)$ and $c \geq 0$. Therefore, by the saddle point theorem, for each $x \in X \setminus \{0\}$, there is a Lagrange multiplier $\lambda(x) \geq 0$ such that for all $\lambda \geq 0$, for all $c \geq 0$ and for all $y \in \Omega_x$ it holds that

$$\begin{aligned} & u(\sigma(x), x) + \rho v(h(x)) + \lambda\left[R(x, h(x)) - (n-1)\sigma(x) - \sigma(x)\right] \\ & \quad \geq u(\sigma(x), x) + \rho v(h(x)) + \lambda(x)\left[R(x, h(x)) - (n-1)\sigma(x) - \sigma(x)\right] \\ & \quad \geq u(c, x) + \rho v(y) + \lambda(x)\left[R(x, y) - (n-1)\sigma(x) - c\right]. \end{aligned} \tag{20}$$

We now define the function $U : \mathbb{R}_+ \times X \mapsto \mathbb{R}$ by

$$U(c, x) = u(c/n, x) + \lambda(x)(n-1)\left[c/n - \sigma(x)\right].$$

Since u is strictly increasing with respect to consumption and $\lambda(x) \geq 0$, it follows that U is also strictly increasing with respect to c. As argued at the beginning of Section 3, it is sufficient to prove that h is rationalized by the dynamic optimization problem $(X, \Omega, \tilde{R}, \rho)$, where the function $\tilde{R} : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$ is defined by

$$\tilde{R}(x, y) = U(R(x, y), x).$$

To show this we first prove the following auxiliary result.

Lemma 2 *Under Assumptions 1–4, the functional equation*

$$v(x) = \tilde{R}(x, h(x)) + \rho v(h(x)) = \max_y\{\tilde{R}(x, y) + \rho v(y) \mid (x, y) \in \Omega\} \tag{21}$$

holds for all $x \in X \setminus \{0\}$.

Proof 2: Assume $x \in X \setminus \{0\}$ and note that $u(\sigma(x), x) = U(n\sigma(x), x)$. From (20) it follows that whenever $\sigma(x)$ and $h(x)$ satisfy $\sigma(x) \geq 0$, $h(x) \in \Omega_x$, and $R(x, h(x)) - \sigma(x) \geq 0$, then the Lagrange multiplier $\lambda(x)$ is such that

$$\begin{aligned} & U(n\sigma(x), x) + \rho v(h(x)) + \lambda\left[R(x, h(x)) - n\sigma(x)\right] \\ & \quad \geq U(n\sigma(x), x) + \rho v(h(x)) + \lambda(x)\left[R(x, h(x)) - n\sigma(x)\right] \\ & \quad \geq U(nc, x) + \rho v(y) + \lambda(x)\left[R(x, y) - nc\right]. \end{aligned}$$

Because the saddle point condition holds, we have for all $c \geq 0$ and all $y \in \Omega_x$ satisfying $R(x, y) - nc \geq 0$ *that* $U(n\sigma(x), x) + \rho v(h(x)) \geq U(nc, x) + \rho v(y)$. Finally, $u(\sigma(x), x) = U(n\sigma(x), x)$ implies $v(x) = \tilde{R}(x, h(x)) + \rho v(h(x))$. Therefore, we have

$$\begin{aligned} v(x) &= \tilde{R}(x, h(x)) + \rho v(h(x)) \\ &= U(n\sigma(x), x) + \rho v(h(x)) \\ &= \max_{c,y} \{U(nc, x) + \rho v(y) \mid c \geq 0,\ y \in \Omega_x,\ R(x, y) - nc \geq 0\}, \end{aligned}$$

which is equivalent to (21). □

To complete the proof of Theorem 2, first note that, since v is concave and $\tilde{R}$ is concave in the second argument, along the equilibrium path $(h^t(x))_{t=0}^{\infty}$ there is a sequence of support prices $(p_t)_{t=0}^{\infty}$ such that, for all $t \geq 1$, $p_t \in \partial v(h^t(x))$ and

$$\tilde{R}(h^{t-1}(x), h^t(x)) - p_{t-1}h^{t-1}(x) + \rho p_t h^t(x) \geq \tilde{R}(y, z) - p_{t-1}y + \rho p_t z$$
$$\text{for all } (y, z) \in \Omega$$

holds.[12] From Assumption 3 it follows that $p_t \geq 0$. Then, by Assumption 5,

$$\begin{aligned} &\sum_{t=1}^{\infty} \rho^{t-1} \tilde{R}(h^{t-1}(x), h^t(x)) - \liminf_{T\to\infty} \sum_{t=1}^{T} \rho^{t-1} \tilde{R}(x_{t-1}, x_t) \\ &\quad \geq \limsup_{T\to\infty} (\rho^T p_T x_T - \rho^T p_T h^T(x)) \geq 0 \end{aligned}$$

holds for all feasible paths $(x_{t-1})_{t=1}^{\infty}$ starting from $x_0 = x \in X\backslash\{0\}$. Therefore, $(h^t(x))_{t=0}^{\infty}$ is an optimal path for the optimization problem $(X, \Omega, \tilde{R}, \rho)$.

Proof of Theorem 3 Since Ω is convex and compact, R and U are continuous and concave, and $\delta \in (0, 1)$, it follows that $(1, X, \Omega, R, U, \delta)$ has an optimal solution. Let g: $X \mapsto [0, 1]$ be the equilibrium strategy; that is, $c_t = g(x_{t-1})$ holds for all t along the optimal solution. Because U is increasing, this implies that $c_t = g(x_{t-1}) = x_{t-1} - x_t^{1/\alpha}$ and, because the model rationalizes the state dynamics h, it follows that $g(x) = x - h(x)^{1/\alpha}$. Substituting the functional form of h stated in (9) and solving for $g(x)$ we obtain $g(x) = \gamma x$, where $\gamma = 2(1 - \alpha\rho)/(2 - \alpha\rho) \in (0, 1)$.

The Euler equation of $(1, X, \Omega, R, U, \delta)$, in contrast, can be written as

$$U'(c_t)R_2(x_{t-1}, x_t) + \delta U'(c_{t+1})R_1(x_t, x_{t+1}) = 0,$$

where R_1 and R_2 are the partial derivatives of the return function with respect to its first and second argument, respectively. Exploiting $R(x, y) = x - y^{1/\alpha}$ and $c_t = \gamma x_{t-1}$ this Euler

[12] See McKenzie (1986).

equation can also be written in the form

$$U'(\gamma x) = \frac{\alpha\delta}{(1-\gamma)^{1-\alpha}x^{1-\alpha}}U'(\gamma(1-\gamma)^{\alpha}x^{\alpha})$$

or, more compactly, as

$$S(c) = \lambda S(\theta c^{\alpha}). \tag{22}$$

Here, $S(c) = cU'(c)$, $\theta = \gamma^{1-\alpha}(1-\gamma)^{\alpha}$ and $\lambda = \alpha\delta/(1-\gamma)$. Note that S is continuous and that $\theta \in (0, 1)$. Iterating equation (22) it follows that

$$S(1) = \lambda S(\theta) = \lambda^2 S(\theta^{1+\alpha}) = \cdots = \lambda^{m+1} S(\theta^{1+\alpha+\cdots+\alpha^m})$$

holds for all $m \in \mathbb{N}$. Because of $\lim_{m\to+\infty} \theta^{1+\alpha+\cdots+\alpha^m} = \theta^{1/(1-\alpha)} > 0$, the above equation can only hold for all m if $\lambda = 1$. By the definition of λ this implies $\gamma = 1 - \alpha\delta$ and, hence, $\delta = \rho/(2-\alpha\rho)$. It remains to be shown that $U(c) = A + B\ln c$ for some real numbers A and $B > 0$.

From equation (22) and $\lambda = 1$ it follows that $S(c) = S(\theta c^{\alpha})$ for all $c > 0$. Differentiating with respect to c and multiplying the resulting equation by c it follows that $F(c) = \alpha F(\theta c^{\alpha})$, where $F(c) = S'(c)c$. Note that this implies

$$F(c) = \alpha F(\theta c^{\alpha}) = \alpha^2 F(\theta^{1+\alpha} c^{\alpha^2}) = \cdots = \alpha^{m+1} F(\theta^{1+\alpha+\cdots+\alpha^m} c^{\alpha^{m+1}})$$

for all $m \in \mathbb{N}$. Because $\lim_{m\to+\infty} \theta^{1+\alpha+\cdots+\alpha^m} = \theta^{1/(1-\alpha)} > 0$ and $\lim_{m\to+\infty}\alpha^{m+1} = 0$ it follows from the above equation that $F(c) = 0$. Since $c > 0$ was arbitrary, we obtain $F(c) = 0$ for all $c > 0$ which, in turn, implies that $S'(c) = 0$ for all $c > 0$. However, $S'(c) = U'(c) + cU''(c)$ and it is well known that any twice continuously differentiable function U for which $U'(c) + cU''(c) = 0$ holds for all $c > 0$ must be an increasing affine transformation of the logarithmic utility function $u(c) = \ln c$. This completes the proof.

Proof of Theorem 4 Because each player's consumption in period t in the game $(n, X, \Omega, R, u, \rho)$ is given by $R(x_{t-1}, x_t) - (n-1)\sigma(x_{t-1})$, the Euler equation for periods t and $t+1$ is

$$u'(c_t)R_2(x_{t-1}, x_t) + \rho u'(c_{t+1})[R_1(x_t, x_{t+1}) - (n-1)\sigma'(x_t)] = 0.$$

Rewrite this equation as

$$R_2(x_{t-1}, x_t) + \frac{\rho u'(c_{t+1})}{u'(c_t)}[R_1(x_t, x_{t+1}) - (n-1)\sigma'(x_t)] = 0.$$

Since h is rationalized by the dynamic optimization problem $(1, X, \Omega, R, u, \delta)$, it follows that the path $(x_t)_{t=0}^{\infty}$ also satisfies the Euler equation of $(1, X, \Omega, R, u, \delta)$, which is given by

$$u'(nc_t)R_2(x_{t-1}, x_t) + \delta u'(nc_{t+1})R_1(x_t, x_{t+1}) = 0.$$

This equation can be rewritten as

$$R_2(x_{t-1}, x_t) + \frac{\delta u'(nc_{t+1})}{u'(nc_t)} R_1(x_t, x_{t+1}) = 0.$$

Because it holds for the isoelastic utility function that $u'(nc_{t+1})/u'(nc_t) = u'(c_{t+1})/u'(c_t)$, it follows from the two Euler equations that

$$(\rho - \delta)\frac{u'(c_{t+1})}{u'(c_t)} R_1(x_t, x_{t+1}) = (n-1)\sigma'(x_t)\frac{\rho u'(c_{t+1})}{u'(c_t)} > 0,$$

where we have made use of $R_1(x, y) > 0$ for all $(x, y) \in \Omega$ and $\sigma'(x_t) > 0$. Therefore, it must hold that $\rho > \delta$.

Proof of Corollary 1 Since $\sigma(x) = n^{-1}R(x, h(x))$ holds for all $x \in X$, we have $\sigma'(\bar{x}) = n^{-1}[R_1(\bar{x}, \bar{x}) + R_2(\bar{x}, \bar{x})h'(\bar{x})]$. In contrast, from the Euler equation of the optimization problem at the steady state, we have $R_2(\bar{x}, \bar{x}) = -\delta R_1(\bar{x}, \bar{x}) < 0$. Combine these results to get $\sigma'(\bar{x}) = n^{-1} R_1(\bar{x}, \bar{x})[1 - \delta h'(\bar{x})] > 0$. The result follows now from Theorem 4.

References

Boldrin, M., and L. Montrucchio (1986), "On the indeterminacy of capital accumulation paths," *Journal of Economic Theory* **40**, 26–39.

Dana, R.-A., and L. Montrucchio (1986), "Dynamic complexity in duopoly games," *Journal of Economic Theory* **40**, 40–56.

Deneckere, R., and S. Pelikan (1986), "Competitive chaos," *Journal of Economic Theory* **40**, 13–25.

Dockner, E., S. Jørgensen, N.V. Long, and G. Sorger (2000), *Differential Games in Economics and Management Science*, New York: Cambridge University Press.

Dutta, P. K., and R. K. Sundaram (1993), "The tragedy of the commons?," *Economic Theory* **3**, 413–26.

Kamihigashi, T. (2003), "Necessity of transversality conditions for stochastic problems," *Journal of Economic Theory* **109**, 140–9.

Kurz, M. (1968), "Optimal economic growth and wealth effects," *International Economic Review* **9**, 348–57.

Levhari, D., and L. J. Mirman (1980), "The great fish war: An example using a dynamic Cournot-Nash solution," *Bell Journal of Economics* **11**, 322–44.

McKenzie, L. W. (1986), "Optimal economic growth, turnpike theorem and comparative dynamics," K. J. Arrow, and M. D. Intriligator, eds, *Handbook of Mathematical Economics 3*, 1281–355, Amsterdam: North-Holland.

Mitra, T., and G. Sorger (1999), "Rationalizing policy functions by dynamic optimization," *Econometrica* **67**, 375–92.

Roy, S. (2010), "On sustained economic growth with wealth effects," *International Journal of Economic Theory* **6**, 29–45.

Samuelson, P. A. (1973), "Optimality of profit, including prices under ideal planning," *Proceedings of the National Academy of Sciences* **70**, 2109–111.

Sorger, G. (2006), "Rationalizability in optimal growth theory," R.-A. Dana, C. Le Van, T. Mitra, and K. Nishimura, eds, *Handbook on Optimal Growth* **1**, 85–113, Berlin: Springer.

Stokey, N. L., and R. E. Lucas (with E.C. Prescott), (1989), *Recursive Methods in Economic Dynamics*, Cambridge: Harvard University Press.